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On the Distribution of the Sums, Products, Quotient and Reliability measure of Lomax Distributed Random Variables Based on FGM Copula

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ABSTRACT
In this article, a Lomax distribution (Pareto Type II) based on Farlie-Gumbel-Morgenstern copula is introduced. Derivations of exact distribution $R = X + Y$, $V = XY$ and $Z = X/(X + Y)$ are obtained in closed form. Corresponding moment properties of these distributions are also derived. The expressions turn out to involve known special functions. Finally, we calculate the closed-form expression of $P(X < Y)$ which is the reliability measure of a component.

Key words: Lomax distribution; Gauss Hypergeometric function; sum, products, quotient of random variables; reliability measure.

1 Introduction

Copula from the latin word *copulare* means to connect or to join (Sklar, 1959). Essentially, copulas' are functions that join or "couple" multivariate distributions to their one-dimensional marginal distribution functions (Nelsen, 1999). Its sole purpose is to describe the interdependence of several random variables (Schmidt, 2006). A copula is a joint distribution function of the uniform marginals (Nelsen, 2003). When marginals are uniform, they are independent. This implies a flat probability density function and any deviation will indicate dependency (Hutchinson and Lai, 2009).

To date, there has been growing interest in copula owing to its usefulness and popularity though not exempt of criticism (Mikosh, 2006). A listing of copula can be found in Hutchinson and Lai (2009), Joe (1997, ch. 5), and Nelsen (2006: 116-119).

In this study, a Farlie-Gumbel-Morgenstern (FGM) copula is considered in constructing a bivariate pdf that accounts dependence between two random variables. Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of the random variables X and Y, respectively, and θ,

$-1 < \theta < 1$, then the joint probability density function or FGM copula density of X and Y is given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1 + \theta\left(2F_X(x) - 1\right)\left(2F_Y(y) - 1\right)\right] \tag{1}$$

where $f_X(x)$ and $f_Y(y)$ are the pdf's of random variable X and Y, respectively. The parameter θ is known as the dependence parameter of X and Y.

The FGM copula was first proposed by Morgenstern (1956). According to Trivedi and Zimmer (2007) it is a perturbation of the product copula. It is also attractive due to its simplicity and tractability. Observe that when θ in (1) equals zero, FGM copula collapses to independence. However, FGM copula is restrictive in the sense that dependency of two marginals should be modest in magnitude (Mukherjee et al., 2012). An extensive applications on FGM with varying marginals can be found in Hutchinson and Lai (2009, ch. 2).

Nadarajah (2005) similar to their other works (Nadarajah & Espejo, 2006; Nadarajah & Kotz, 2007) concern on obtaining exact distributions on the sum, product and quotient of some known bivariate distributions. For the evaluation of $P(X < Y)$, the works of Kotz et al. (2003) gives a comprehensive account. Also, the recent work of Domma and Giordano (2013) provide a good survey on this matter. We emphasized that most of works done assumed that X and Y are either independent or correlated. In reality, a bivariate distribution often admits a certain specific form of dependence between margins and using copula-based approach is an advantage.

In this note, a bivariate Lomax distribution also called the Pareto type II distribution constructed from FGM copula is introduced. As to our knowledge, there is still no research done with this marginal.

The paper is organized as follows. Section 2 is devoted on derivations of explicit expressions for the pdfs of $R = X + Y$, $V = XY$ and $Z = X/(X + Y)$, resp. while section 3 is devoted in derivation of raw moments of all pdfs obtained in section 2. Finally, section 4 closed this article with the calculation of $P(X < Y)$.

The calculations of this paper involve several special functions. These include the incomplete beta function

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,$$

and, the Gauss Hypergeometric function

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The following results which can be found in Nadarajah and Espejo (2006) are needed in the subsequent discussions.

LEMMA 1. *For any $\rho > \alpha > 0$,*

$$\int_0^\infty \frac{s^{\alpha-1}}{(s+z)^\rho} ds = z^{\alpha-\rho} B(\alpha, \rho-\alpha), \quad z \in \mathbb{R}, \tag{2}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

for $a > 0$ and $b > 0$ is the beta function.

LEMMA 2. *For $0 < \alpha < \rho + \lambda$,*

$$\begin{aligned} &\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda} dx \\ &\qquad = z^{-\lambda} y^{\alpha-\rho} B(\alpha, \rho+\lambda-\alpha) {}_2F_1\left(\alpha, \lambda; \rho+\lambda; 1-\frac{y}{z}\right). \end{aligned} \tag{3}$$

LEMMA 3. *For $p > 0$ and $q > 0$,*

$$\begin{aligned} &\int_a^b (x-a)^{p-1}(b-x)^{q-1}(cx+d)^r dx \\ &\qquad = (b-a)^{p+q-1}(ac+d)^r B(p,q) {}_2F_1\left(p, -r; p+q; \frac{c(a-b)}{ac+d}\right). \end{aligned} \tag{4}$$

2 Pdfs

Let X and Y be two independent Lomax distributed random variables with probability density functions (pdf) given by

$$f_X(x;\alpha,\theta) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}; \quad x > 0,\, \alpha > 0,\, \theta > 0 \tag{5}$$

and

$$f_Y(y;\alpha,\theta) = \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}}; \quad y > 0,\, \alpha > 0,\, \theta > 0, \tag{6}$$

respectively.

The cumulative distribution functions (cdf) of X and Y are known to be

$$F_X(x;\alpha,\theta) = 1 - \Big(\frac{\theta}{x+\theta}\Big)^{\alpha}; \quad x>0,\, \alpha>0,\, \theta>0 \tag{7}$$

and

$$F_Y(y;\alpha,\theta) = 1 - \Big(\frac{\theta}{y+\theta}\Big)^{\alpha}; \quad y>0,\, \alpha>0,\, \theta>0, \tag{8}$$

respectively.

The following result is the joint pdf derived from FGM copula using Lomax distribution as marginals. It will be used often in this paper as our random variables X and Y follows this joint density.

Theorem 2.1. *Let X and Y be random variables that follows Lomax distribution with pdfs in (5) and (6) and cdfs in (7) and (8), respectively. Then the joint density function is given by*

$$f_{X,Y}(x,y;\alpha,\theta;\rho) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} \frac{\alpha\theta^{\alpha}}{(y+\theta)^{\alpha+1}} \left[1+\rho\left(2\Big(\frac{\theta}{x+\theta}\Big)^{\alpha}-1\right)\left(2\Big(\frac{\theta}{y+\theta}\Big)^{\alpha}-1\right)\right] \tag{9}$$

where $x,\ y,\ \alpha,\ \theta$ are all positive and $|\rho| \leq 1$.

Proof. Plugging-in equations (5)–(6) in the FGM copula, we have

$$f_{X,Y}(x,y;\alpha,\theta;\rho) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} \frac{\alpha\theta^{\alpha}}{(y+\theta)^{\alpha+1}} \left[1+\rho\left(2\Big(\frac{\theta}{x+\theta}\Big)^{\alpha}-1\right)\left(2\Big(\frac{\theta}{y+\theta}\Big)^{\alpha}-1\right)\right].$$

It can be shown that (9) is nonnegative. Now to show that $\int_0^\infty \int_0^\infty f_{X,Y}\,(x,y;\alpha,\theta;\rho)\,dxdy$ is unity. Consider the following

$$\begin{aligned}\int_0^\infty \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} \left[2\left(\frac{\theta}{x+\theta}\right)^{\alpha}-1\right]dx &= \int_0^\infty \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} \left[1-2\left(1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}\right)\right]dx \\ &= 1 - \int_0^\infty \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} 2\left(1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}\right)dx.\end{aligned}$$

Let $u = 1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$. Then $du = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}dx$. If $x=0$, then $u=0$. As $x\to\infty$, $u\to 1$. Hence,

$$1 - \int_0^\infty \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} 2\left(1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}\right)dx = 1 - \int_0^1 2u\,du = 0.$$

Thus,

$$\int_0^\infty \int_0^\infty \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} \rho \left[2\left(\frac{\theta}{x+\theta}\right)^\alpha - 1\right]\left[2\left(\frac{\theta}{y+\theta}\right)^\alpha - 1\right] dxdy = 0.$$

Also

$$\int_0^\infty \int_0^\infty \frac{(\alpha\theta^\alpha)^2}{[(x+\theta)(y+\theta)]^{\alpha+1}} dxdy = 1.$$

Consequently, we have

$$\int_0^\infty \int_0^\infty f_{X,Y}(x, y; \alpha, \theta; \rho)\, dxdy = 1.$$

□

The following figure illustrates the pdf in (9) for specific values: $\alpha = .12$, $\theta = 2$, $\rho = 0.5$.

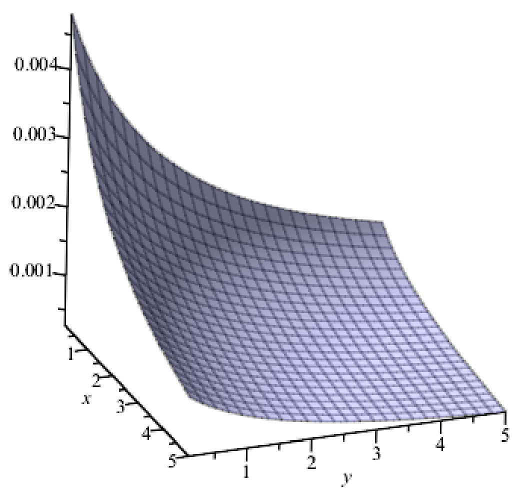

Figure 1: Graph of the pdf in (9)

Theorems (2.2)–(2.5) derive the pdfs of $R = X + Y$, $V = XY$ and $W = X/(X + Y)$ when X and Y are distributed according to (9). In the subsequent, we assume that α, θ are positive real numbers and $\rho \in [-1, 1]$.

Theorem 2.2. *If X and Y are jointly distributed according to (9), then the density function of $V = XY$ is*

$$\begin{aligned} f_V(v;\alpha,\theta;\rho) = (\alpha\theta^\alpha)^2 \Bigg[& \frac{(1+\rho)}{v^{\alpha+1}} B(\alpha+1,\alpha+1)\, {}_2F_1\left(\alpha+1,\alpha+1;2\alpha+2;1-\frac{\theta^2}{v}\right) \\ & + \frac{4\rho\theta^{2\alpha}}{v^{2\alpha+1}} B(2\alpha+1,2\alpha+1)\, {}_2F_1\left(2\alpha+1,2\alpha+1;4\alpha+2;1-\frac{\theta^2}{v}\right) \\ & - \frac{2\rho}{v^{\alpha+1}} B(\alpha+1,2\alpha+1)\, {}_2F_1\left(\alpha+1,\alpha+1;3\alpha+2;1-\frac{\theta^2}{v}\right) \\ & - \frac{2\rho\theta^{2\alpha}}{v^{2\alpha+1}} B(2\alpha+1,\alpha+1)\, {}_2F_1\left(2\alpha+1,2\alpha+1;3\alpha+2;1-\frac{\theta^2}{v}\right) \Bigg] \end{aligned} \tag{10}$$

for $0 < v < \infty$.

Proof. From (9), the joint pdf of $(X,Y) = \left(X, \frac{V}{X}\right)$ can be expressed as

$$\begin{aligned} f_{X,V}\left(x,\frac{v}{x};\alpha,\theta;\rho\right) = (\alpha\theta^\alpha)^2 \Bigg\{ & \frac{1+\rho}{\left[(x+\theta)(\frac{v}{x}+\theta)\right]^{\alpha+1}} + \frac{4\rho\theta^{2\alpha}}{\left[(x+\theta)(\frac{v}{x}+\theta)\right]^{2\alpha+1}} \\ & - \frac{2\rho\theta^\alpha}{(x+\theta)^{2\alpha+1}\left(\frac{v}{x}+\theta\right)^{\alpha+1}} - \frac{2\rho\theta^\alpha}{(x+\theta)^{\alpha+1}\left(\frac{v}{x}+\theta\right)^{2\alpha+1}} \Bigg\}. \end{aligned}$$

By Rohatgi's well-known result (1976, p. 141), the pdf of $V = XY$ becomes

$$f_V(v;\alpha,\theta;\rho) = (\alpha\theta^\alpha)^2 \left[(1+\rho)A(1,1) + 4\rho\theta^{2\alpha}A(2,2) - 2\rho\theta^\alpha A(2,1) - 2\rho\theta^\alpha A(1,2)\right] \tag{11}$$

where

$$A(h,k) = \int_0^\infty x^{k\alpha}(x+\theta)^{-(h\alpha+1)}(v+\theta\cdot x)^{-(k\alpha+1)}\,dx, \quad \text{for } h,k \in \{1,2\}.$$

Using Lemma (2) we obtain

$$A(h,k) = \theta^{k\alpha-h\alpha}v^{-(k\alpha+1)}B(k\alpha+1,h\alpha+1)\,{}_2F_1\left(k\alpha+1,k\alpha+1;(h+k)\alpha+2;1-\frac{\theta^2}{v}\right). \tag{12}$$

Applying (12) to the equation (11) will result to (10). □

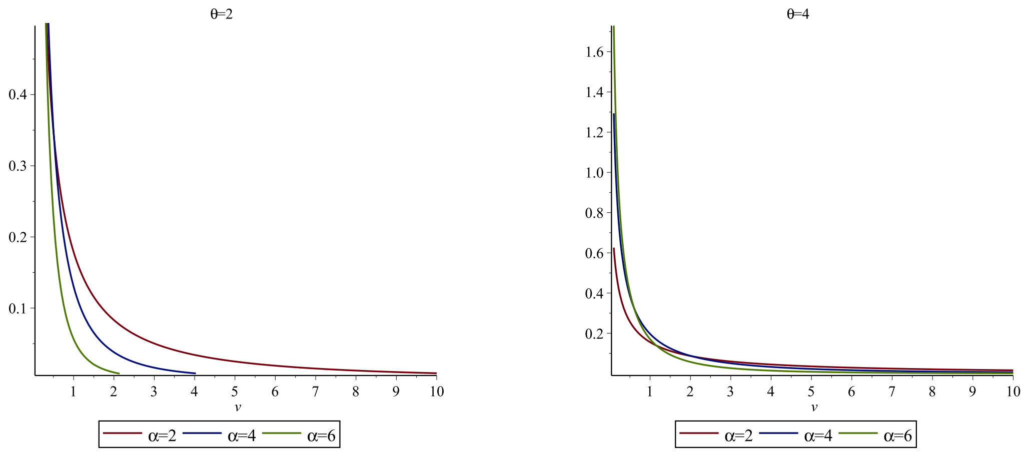

Figure 2: Graph of the pdf in (10) with selected values of θ and α.

Figure 2 illustrate the shape of the pdf in (10) for $\theta = 2, 4$. Each plot contains three curves corresponding to selected values of α. The effect of the parameters is evident.

Theorem 2.3. *If X and Y are jointly distributed according to (9), then the distribution of $W = \frac{X}{Y}$ is*

$$\begin{aligned} f_W(w;\alpha,\theta;\rho) = (\alpha)^2 \Big[&(1+\rho)B(2,2\alpha)\ {}_2F_1\left(2,\alpha+1;2\alpha+2;1-w^{-1}\right) \\ &+ 4\rho B(2,4\alpha)\ {}_2F_1\left(2,2\alpha+1;4\alpha+2;1-w^{-1}\right) \\ &- 2\rho B(2,3\alpha)\ {}_2F_1\left(2,\alpha+1;3\alpha+2;1-w^{-1}\right) \\ &- 2\rho B(2,3\alpha)\ {}_2F_1\left(2,2\alpha+1;3\alpha+2;1-w^{-1}\right)\Big]. \end{aligned} \tag{13}$$

for $0 < w < \infty$.

Proof. From (9), the joint pdf of $(X,Y) = \left(X, \frac{X}{W}\right)$ can be expressed as

$$\begin{aligned} f_{X,W}\left(x,\frac{x}{w};\alpha,\theta;\rho\right) = (\alpha\theta^\alpha)^2 \Bigg\{ &\frac{1+\rho}{\left[(x+\theta)(\frac{x}{w}+\theta)\right]^{\alpha+1}} + \frac{4\rho\theta^{2\alpha}}{\left[(x+\theta)(\frac{x}{w}+\theta)\right]^{2\alpha+1}} \\ &- \frac{2\rho\theta^\alpha}{(x+\theta)^{2\alpha+1}\left(\frac{x}{w}+\theta\right)^{\alpha+1}} - \frac{2\rho\theta^\alpha}{(x+\theta)^{\alpha+1}\left(\frac{x}{w}+\theta\right)^{2\alpha+1}} \Bigg\}. \end{aligned}$$

By Rohatgi's result, the pdf of $W = \frac{X}{Y}$ can be expressed as

$$f_W(w;\alpha,\theta;\rho) = (\alpha\theta^{\alpha})^2 \left[(1+\rho)\, C(1,1) + 4\rho\theta^{2\alpha} C(2,2) - 2\rho\theta^{\alpha} C(2,1) - 2\rho\theta^{\alpha} C(1,2)\right] \quad (14)$$

where

$$C(h,k) = \int_0^{\infty} w^{k\alpha+1} x\,(x+\theta)^{-(h\alpha+1)}\,(x+\theta\cdot w)^{-(k\alpha+1)}\,dx. \quad (15)$$

for $h,k \in \{1,2\}$.

Using Lemma (2) one can get

$$C(h,k) = \theta^{-(h+k)\alpha} B\left(2,(h+k)\alpha\right)\ {}_2F_1\left(2, k\alpha+1; (h+k)\alpha+2; 1-w^{-1}\right). \quad (16)$$

By (16), the following terms in (14) are obvious.

(1) $(1+\rho)\, C(1,1) = (1+\rho)\theta^{-2\alpha} B(2,2\alpha)\ {}_2F_1\left(2,\alpha+1;2\alpha+2;1-w^{-1}\right);$

(2) $4\rho\theta^{2\alpha} C(2,2) = 4\rho\theta^{-2\alpha} B(2,4\alpha)\ {}_2F_1\left(2,2\alpha+1;4\alpha+2;1-w^{-1}\right);$

(3) $-2\rho\theta^{\alpha} C(2,1) = -2\rho\theta^{-2\alpha} B(2,3\alpha)\ {}_2F_1\left(2,\alpha+1;3\alpha+2;1-w^{-1}\right);$

(4) $-2\rho\theta^{\alpha} C(1,2) = -2\rho\theta^{-2\alpha} B(2,3\alpha)\ {}_2F_1\left(2,2\alpha+1;3\alpha+2;1-w^{-1}\right);$

The result follows by using items (1)–(4) in (14). □

Theorem 2.4. *If X and Y are jointly distributed according to (9), then the distribution of $Z = \frac{X}{X+Y}$ is*

$$\begin{aligned} f_Z(z;\alpha,\theta;\rho) = \alpha^2 \Bigg[& (1+\rho) B(2,2\alpha)\ {}_2F_1\left(2,\alpha+1;2\alpha+2;\frac{2z-1}{z}\right) \\ & + 4\rho B(2,4\alpha)\ {}_2F_1\left(2,2\alpha+1;4\alpha+2;\frac{2z-1}{z}\right) \\ & - 2\rho B(2,3\alpha)\ {}_2F_1\left(2,\alpha+1;3\alpha+2;\frac{2z-1}{z}\right) \\ & - 2\rho B(2,3\alpha)\ {}_2F_1\left(2,2\alpha+1;3\alpha+2;\frac{2z-1}{z}\right) \Bigg] \end{aligned} \quad (17)$$

for $0 < z < 1$.

Proof. Consider the transformation: $(X,Y) \longrightarrow (R,Z) = \left(X+Y, \frac{X}{X+Y}\right)$ so that

$$f_{R,Z}(r,z;\alpha,\theta;\rho) = (\alpha\theta^{\alpha})^2 \left\{ \frac{1+\rho}{\left[(rz+\theta)(r-rz+\theta)\right]^{2\alpha+1}} + \frac{4\rho\theta^{2\alpha}}{\left[(rz+\theta)(r-rz+\theta)\right]^{2\alpha+1}} - \frac{2\rho\theta^{\alpha}}{(rz+\theta)^{2\alpha+1}(r-rz+\theta)^{\alpha+1}} - \frac{2\rho\theta^{\alpha}}{(rz+\theta)^{\alpha+1}(r-rz+\theta)^{2\alpha+1}} \right\}$$

Note that the jacobian of transformation is r, thus

$$f_Z(z;\alpha,\theta;\rho) = (\alpha\theta^{\alpha})^2 \left\{ (1+\rho) D(1,1) + 4\rho\theta^{2\alpha} D(2,2) - 2\rho\theta^{\alpha} D(2,1) - 2\rho\theta^{\alpha} D(1,2) \right\} \tag{18}$$

where

$$D(h,k) = \int_0^{\infty} r (rz+\theta)^{-(h\alpha+1)} (r-rz+\theta)^{-(k\alpha+1)} \, dr \tag{19}$$

for $h,k \in \{1,2\}$.
Let $u = (1-z)r$. Then $dr = \frac{1}{1-z} du$. One can obtain $D(h,k)$ as follows

$$D(h,k) = \int_0^{\infty} \frac{u}{1-z} \left[\frac{uz}{1-z} + \theta\right]^{-(h\alpha+1)} [u+\theta]^{-(k\alpha+1)} \frac{1}{1-z} du. \tag{20}$$

Using Lemma (2), we have

$$D(h,k) = \theta^{-(k+h)\alpha} B(2,(h+k)\alpha)_2F_1\left(2, k\alpha+1; (h+k)\alpha+2; \frac{2z-1}{z}\right) \tag{21}$$

Combining (21) and (18) the result in (17) follows. □

The next figure illustrates the pdf in (17) for specific values: $\rho = 0.5, \alpha = 2, 4,$ and 6. Note that $\frac{X}{X+Y}$ is between 0 and 1. The graph shows the domain on $[0,1]$.

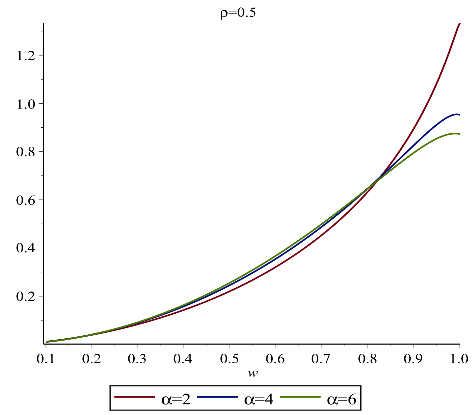

Figure 3: Graph of the pdf in (17)

Theorem 2.5. *If X and Y are jointly distributed according to (9) then the density function of $R = X + Y$ is given by*

$$f_R(r;\alpha,\theta;\rho) = r(\alpha\theta^\alpha)^2\Bigg\{(1+\rho)\,\theta^{-(\alpha+1)}(r+\theta)^{-(\alpha+1)}\sum_{j=1}^{\infty}\left[\binom{\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}A\right]$$
$$+4\rho\theta^{2\alpha}\theta^{-(2\alpha+1)}(r+\theta)^{-(2\alpha+1)}\sum_{j=1}^{\infty}\left[\binom{2\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}B\right]$$
$$-2\rho\theta^{\alpha}\theta^{-(\alpha+1)}(r+\theta)^{-(\alpha+1)}\sum_{j=1}^{\infty}\left[\binom{2\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}A\right]$$
$$-2\rho\theta^{\alpha}\theta^{-(2\alpha+1)}(r+\theta)^{-(2\alpha+1)}\sum_{j=1}^{\infty}\left[\binom{\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}B\right]\Bigg\}$$

where $A = {}_2F_1\left(j,\alpha+1;j+1;\frac{r}{r+\theta}\right)$ *and* $B = {}_2F_1\left(j,2\alpha+1;j+1;\frac{r}{r+\theta}\right)$ *for* $0 < r < \infty$.

Proof. Consider the transformation: $(X,Y) \longrightarrow (R,Z) = \left(X+Y, \frac{X}{X+Y}\right)$ so that

$$f_{R,Z}(r,z;\alpha,\theta;\rho) = (\alpha\theta^\alpha)^2\Bigg\{\frac{1+\rho}{[(rz+\theta)(r-rz+\theta)]^{2\alpha+1}} + \frac{4\rho\theta^{2\alpha}}{[(rz+\theta)(r-rz+\theta)]^{2\alpha+1}}$$
$$-\frac{2\rho\theta^\alpha}{(rz+\theta)^{2\alpha+1}(r-rz+\theta)^{\alpha+1}} - \frac{2\rho\theta^\alpha}{(rz+\theta)^{\alpha+1}(r-rz+\theta)^{2\alpha+1}}\Bigg\}$$

The jacobian of transformation is r, thus

$$f_R(r;\alpha,\theta;\rho) = r(\alpha\theta^\alpha)^2\left\{(1+\rho)\,G(1,1) + 4\rho\theta^{2\alpha}G(2,2) - 2\rho\theta^\alpha G(2,1) - 2\rho\theta^\alpha G(1,2)\right\}$$

$$(22)$$

where

$$G(h,k) = \int_0^1 (rz+\theta)^{-(h\alpha+1)} (r-rz+\theta)^{-(k\alpha+1)}\, dz \tag{23}$$

for $h, k \in \{1, 2\}$.
Using Lemma (3), one can obtain $G(h,k)$ as follows

$$\begin{aligned}
G(h,k) &= \theta^{-(h\alpha+1)} \sum_{j=0}^{\infty} \Bigg[\binom{h\alpha+j}{j} \left(-\frac{r}{\theta}\right)^{j} \int_0^1 z^j (-rz+r+\theta)^{-(k\alpha+1)}\, dz \Bigg] \\
&= \theta^{-(h\alpha+1)} \sum_{j=1}^{\infty} \Bigg[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} \int_0^1 (z-0)^{j-1} (1-z)^{1-1} \\
&\qquad (-rz+r+\theta)^{-(k\alpha+1)}\, dz \Bigg] \\
&= \theta^{-(h\alpha+1)} \sum_{j=1}^{\infty} \Bigg[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} (r+\theta)^{-(k\alpha+1)} B(j,1) \qquad (24) \\
&\qquad {}_2F_1\left(j, k\alpha+1; j+1; \frac{r}{r+\theta}\right) \Bigg] \\
&= \theta^{-(h\alpha+1)} (r+\theta)^{-(k\alpha+1)} \sum_{j=1}^{\infty} \Bigg[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \\
&\qquad {}_2F_1\left(j, k\alpha+1; j+1; \frac{r}{r+\theta}\right) \Bigg]
\end{aligned}$$

Combining (22) and (24), the result follows immediately. □

3 Moments

Theorem 3.1. *Let X and Y be jointly distributed according to (9). Then the (a,b)-th product moment of bivariate Lomax density function denoted by ${\mu'}_{a,b;\rho}(X,Y)$ is given by*

$$\begin{aligned}
{\mu'}_{a,b;\rho}(X,Y) = \Gamma(a+1)\Gamma(b+1)\theta^{a+b} \Big[&\frac{\Gamma(\alpha-a)\Gamma(\alpha-b)}{\Gamma^2(\alpha)} \\
&+ \rho\Big(\frac{\Gamma(2\alpha-a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a)}{\Gamma(\alpha)}\Big)\Big(\frac{\Gamma(2\alpha-b)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-b)}{\Gamma(\alpha)}\Big)\Big]
\end{aligned} \tag{25}$$

where x, y, α, θ, are all positive, $|\rho| \le 1$ and $\max\{a,b\} < \alpha$.

Proof. By definition, one can expressed the (a, b)-th moment of $f_{X,Y}(x, y; \alpha, \theta; \rho)$ as

$$\begin{aligned}\mu'_{a,b;\rho}(X,Y) = &\int_0^\infty \int_0^\infty x^a y^b \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} dxdy \\ &+ \rho\Bigg[\left(\int_0^\infty \left(2\left(\frac{\theta}{y+\theta}\right)^\alpha - 1\right) \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} y^b dy\right) \\ &\left(\int_0^\infty \left(2\left(\frac{\theta}{x+\theta}\right)^\alpha - 1\right) \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} x^a dx\right)\Bigg].\end{aligned}$$

By Lemma 1, one can show the following integrals:

(1)

$$\int_0^\infty x^a \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx = \alpha\theta^a B(a+1, \alpha - a);$$

(2)

$$\int_0^\infty y^b \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} dy = \alpha\theta^b B(b+1, \alpha - b);$$

(3)

$$\int_0^\infty x^a \frac{\alpha\theta^{2\alpha}}{(x+\theta)^{2\alpha+1}} dx = \alpha\theta^a B(a+1, 2\alpha - a);$$

(3) Finally,

$$\int_0^\infty y^b \frac{\alpha\theta^{2\alpha}}{(y+\theta)^{2\alpha+1}} dy = \alpha\theta^b B(b+1, 2\alpha - b);$$

Then the result follows directly. □

Theorem 3.2. *If X and Y are jointly distributed according to 9, then the a-th raw moment of the random variable V is*

$$\mu'_{a;\rho}(V) = \theta^{2a}\Gamma^2(a+1)\left[\frac{\Gamma^2(\alpha-a)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a)}{\Gamma(\alpha)}\right)^2\right]. \quad (26)$$

Proof. Notice that

$$E(V^a) = E((X \cdot Y)^a) = E(X^a \cdot Y^a).$$

Putting $b = a$ in (25), the result follows. □

We state the next result without proof since the proof is similar to that of Theorem 3.2.

Theorem 3.3. *If X and Y are jointly distributed according to (9), then a-th raw moment of $W = \frac{X}{Y}$ is*

$$\mu'_{a;\rho}(W) = \Gamma(a+1)\Gamma(1-a)\left[\frac{\Gamma(\alpha-a)\Gamma(\alpha+a)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a)}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(2\alpha+a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha+a)}{\Gamma(\alpha)}\right)\right] \tag{27}$$

Theorem 3.4. *If X and Y are jointly distributed according to (9), then the a-th raw moment of $Z = \frac{X}{X+Y}$ is*

$$\mu'_{a,\rho}(Z) = \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\Gamma(a+k+1)\Gamma(1-a-k)\cdot \left[\frac{\Gamma(\alpha-a-k)\Gamma(\alpha+a+k)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a-k)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a-k)}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(2\alpha+a+k)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha+a+k)}{\Gamma(\alpha)}\right)\right] \tag{28}$$

Proof. Notice that

$$\begin{aligned}
E(Z^a) = E\left(X^a\cdot(X+Y)^{-a}\right) &= E\left(\left(\frac{X}{Y}\right)^a\left(\frac{X}{Y}+1\right)^{-a}\right)\\
&= E\left(\left(\frac{X}{Y}\right)^a\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\left(\frac{X}{Y}\right)^k\right)\\
&= E\left(\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\left(\frac{X}{Y}\right)^{a+k}\right)\\
&= \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k E\left(W^{a+k}\right)
\end{aligned}$$

Using Theorem 3.2, the result in (28) follows. □

Theorem 3.5. *If X and Y are jointly distributed according to (9), then the a-th raw* *moment of $R = X + Y$ is*

$$\mu'_{a;\rho}(R) = \theta^a \sum_{i=0}^{a} \binom{a}{i} \Bigg\{ \Gamma(i+1)\Gamma(a-i+1) \left[\frac{\Gamma(\alpha-i)\Gamma(\alpha-(a-i))}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-i)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-i)}{\Gamma(\alpha)}\right) \left(\frac{\Gamma(2\alpha-(a-i))}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-(a-i))}{\Gamma(\alpha)}\right)\right]\Bigg\} \tag{29}$$

Proof. Since $R^a = (X+Y)^a = \sum_{i=0}^{a} \binom{a}{i} X^i \cdot Y^{a-i}$, then

$$\mu'_{a;\rho}(R) = E(R^a) = \sum_{i=0}^{a} \binom{a}{i} E\left(X^i Y^{a-i}\right) = \sum_{i=0}^{a} \binom{a}{i} \mu'_{i,a-i;\rho}\left(X^i Y^{a-i}\right).$$

By putting $a = i$ and $b = a - i$ in (25), the result follows. □

4 Reliability measure Q

In this section, we calculate the reliability measure $Q = P(X < Y)$ suggested by Domma and Giordano (2013). The measure Q is given by

$$Q = P(X < Y) = \int_0^{+\infty} \int_0^{x} f_{X,Y}(x,y)dydx \tag{30}$$

where $f_{X,Y}(x,y)$ is the FGM copula density. Hence, Q can be written in linear form

$$Q = Q_I + \theta D \tag{31}$$

where

$$Q_I = \mathbb{E}_X\left[F_Y(X)\right] \tag{32}$$

and

$$D = \mathbb{E}_X\left\{F_Y(X)\left[1 - F_Y(X)\right]\left[1 - 2F_X(X)\right]\right\}. \tag{33}$$

Note that marginal distributions $F_X(x)$ and $F_Y(y)$, can be nonidentical or need not be in the same family of distribution. In our computation, we consider a nonidentical Lomax distributed marginal, that is, with cumulative distribution functions $F_X(x;\alpha,\theta) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$ and $F_Y(y;\beta,\theta) = 1 - \left(\frac{\theta}{y+\theta}\right)^{\beta}$, the same shape parameter θ is assumed for simplicity; α, β, θ are all strictly positive parameters. Substituting the above cumulative functions in

(32) and (33), after algebra we obtain for $Q_I = \frac{\beta}{\alpha+\beta}$. This is the measure of reliability for *independent* stress and strength variables in the sense that when $\theta = 0$, (31) holds. Calculating for D, we have $D = \frac{\alpha\beta(\beta-\alpha)}{(2\alpha+\beta)(\alpha+\beta)(\alpha+2\beta)}$. D is the coefficient of θ which determines the weight of the dependence.

Our result is related but different to the reliability measure for Burr III stress and strength variables by Domma and Giordano (2013) in the sense that α and β are interchanged. We note further that if X and Y are identically distributed ($\alpha = \beta$) then $Q = 0.5$ regardless of θ. To this end, the measure of reliability $P(X < Y)$ is given by

$$Q = \frac{\beta}{\alpha+\beta} - \theta\frac{\alpha\beta(\alpha-\beta)}{(2\alpha+\beta)(\alpha+\beta)(\alpha+2\beta)}. \tag{34}$$

5 Conclusion

In this paper, we have derived the probability density functions of sum, product and quotient of two random variables both having Lomax distribution. We also derived each corresponding rth raw moments. These moments are useful in the estimation of sum, products or quotients of X and Y. Irregardless of the application setting of random variables, the results are expressed in terms of beta and hypergeometric functions. Hence, one can implement a code as these special functions are readily available in most common software. Finally, we calculate the reliability measure of a component for a given random variables. The expression is direct and simple. It is worth to mention that using FGM copula for accommodating the association for two random variables X and Y applied to sum, product and quotient of X and Y is new. One setback though of using this copula is due to its weak dependence. However, through FGM, we produced fairly simple and elegant results. Furthermore, our choice of using FGM is due to mathematical convenience, hence, we suggests that one can always consider a rather general copula that better capture the association of random variables.

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